Optimal Uncertainty Quantification in Complex Systems

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QMU – Certification viewpoint

- Certification: PoF of the system below tolerance,
  \[ P[\text{failure}] = P[y \notin A] \leq \epsilon \]
- Exact probability of failure:
  \[
  P[\text{failure}] = \int \begin{cases} 
  0, & \text{if } G(x) \in A \\
  1, & \text{if } G(x) \notin A 
  \end{cases} d\mu(x)
  \]
QMU – A simple truss example

- System input: Applied force \( (f) \)
- System output: Tip deflection \( (v) \)
- Response function \( (G) \): Energy minimization, static equilibrium
- Model \( (F) \): Energy minimization with approximate strain-energy density function \( W \)
- Failure criterion: \( v > \text{threshold} \)
- To compute: 
  \[
  P[\text{failure}] = \int \left\{ \begin{array}{ll}
  0, & \text{if } v < v_{\text{max}} \\
  1, & \text{if } v \geq v_{\text{max}}
  \end{array} \right\} d\mu(f)
  \]
QMU – A simple truss example

• Assume: Deterministic response, known probability distribution of inputs

\[ v = G(f) \]
QMU – A simple truss example

- Assume: Stochastic response function, known probability distribution of inputs

\[ \text{cumulative distribution function} \]

\[ 1 - \text{PoF}(\omega) \]

\[ v = G(f, \omega) \]

Unknown unknowns!
QMU – Certification view

- Certification: PoF of the system below tolerance,
  \[ P[\text{failure}] = P[y \notin A] \leq \epsilon \]

- Exact probability of failure:
  \[ P[\text{failure}] = \int \begin{cases} 0, & \text{if } G(x) \in A \\ 1, & \text{if } G(x) \notin A \end{cases} d\mu(x) \]
QMU – Essential difficulties

- Input space of high dimension, unknown unknowns
- Probability distribution of inputs not known in general
- System response stochastic, not known in general
- Models are inaccurate, partially verified & validated
- System performance cannot be tested on demand
- Legacy data incomplete, inconsistent, and noisy…
QMU – Conservative certification

- Conservative certification: **Upper bound** on the PoF of the system below tolerance,
  \[ \mathbb{P}[^{\text{failure}}] = \mathbb{P}[y \notin A] \leq \text{upper bound} \leq \epsilon \]
- Objective: Obtain tight (optimal?) PoF upper bounds from all known information about the system...
Optimal Uncertainty Quantification

- Wanted: $\mathbb{P}[\text{failure}] = \mathbb{E}_\mu\{G \in A\}$
- Assume information about $(\mu, G)$: Data, models...
- Admissible set: $A = \{ (\mu, G) \text{ compatible with info} \}$
- Optimal PoF bounds given $A$:

$$
\inf_{(\mu, G) \in A} \mathbb{E}_\mu(\{G \in A\}) \leq \text{PoF} \leq \sup_{(\mu, G) \in A} \mathbb{E}_\mu(\{G \in A\})
$$
OUQ – The Reduction Theorem

**Theorem** [Owhadi et al. (2011)] Suppose that

\[ A = \left\{ (\mu, G) \mid \begin{array}{c} \langle \text{some conditions on } G \text{ alone} \rangle \\
\mathbb{E}_\mu[\varphi_1] \leq 0, \ldots, \mathbb{E}_\mu[\varphi_n] \leq 0 \end{array} \right\}. \]

Let:

\[ A_{\text{red}} = \left\{ (\mu, G) \in A \mid \mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}, \ \alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = 1 \right\} \]

Then:

\[
\begin{align*}
\inf_{(\mu, G) \in A} \mathbb{E}_\mu(\{G \in A\}) &= \inf_{(\mu, G) \in A_{\text{red}}} \mathbb{E}_\mu(\{G \in A\}) \\ 
\sup_{(\mu, G) \in A} \mathbb{E}_\mu(\{G \in A\}) &= \sup_{(\mu, G) \in A_{\text{red}}} \mathbb{E}_\mu(\{G \in A\})
\end{align*}
\]

- OUQ problem is reduced to optimization over finite-dimensional space of measures: Program feasible!
Example – Seismic risk assessment

Simulation of seismic waves from rupture initiating at Parkfield, central California, and propagating over Los Angeles basin (http://krishnan.caltech.edu/krishnan/res.html)

3D truss structure of power-line tower
Example – Seismic risk assessment

- Ground motion acceleration:
  \[ \ddot{u}_0(t) = (\psi * s)(t) \]
  where: \( s(t) \equiv \text{Source activity} \)
  \( \psi(t) \equiv \text{Transfer function} \)

- Structural response:
  \[ M\ddot{u} + C\dot{u} + Ku = f(t) - MT\ddot{u}_0(t) \]

- Failure criterion: \( G \leq 0 \), where
  \[ G = \min_{i \in \text{members}} \left\{ \sigma_y - \max_{t \geq 0} |\sigma_i(t)| \right\} \]

3D truss structure of power-line tower
Example – Seismic risk assessment

- Assumptions on source term $s(t)$:
  - Piecewise constant in time
  - Random amplitudes in $[-a_{\text{max}}, a_{\text{max}}]$ (given by Richter magnitude $M$) with zero mean
  - Random time interval durations with bounded mean

- Assumptions on transfer function $\psi(t)$:
  - Piecewise linear in time
  - Random amplitudes with zero mean, bounded $L^2$ norm

- Reduced OUQ problem: Global optimization in 179 dimensions

- One PoF calculation takes $O(24 \text{ hours})$ on $O(1000)$ AMD opteron cluster

3D truss structure of power-line tower
Example – Seismic risk assessment

• Assumptions on source term $s(t)$:
  – Piecewise constant (boxcar) in time
  – Random amplitudes in $[-a_{\text{max}}, a_{\text{max}}]$ (given by Richter magnitude M) with zero mean
  – Random time interval durations with bounded mean

• Assumptions on transfer function $\psi(t)$:
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• Reduced OUQ problem: Global optimization in 179 dimensions

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Example – Seismic risk assessment

Optimal PoF upper and lower bounds for steel tower vs. Richter scale magnitude $M$ at hypocentral distance $R=25$ km, ($a_{\text{max}}$ given by Esteva's semi-empirical expression as a function of $M$)
Concluding remarks…

- Rigorous and conservative certification can be achieved by means of PoF upper bounds!
- PoF bounds ‘fold in’ all information available on the system (experimental data, V&V’d physics models…)
- PoF bounds are similar in spirit to bounds on effective moduli of elastic composites (which cannot be obtained exactly in general from existing data on the composite)
- However: Bounds can be suboptimal (e.g., Voigt, Reuss…) and result in excessive conservatism
- It possible to compute optimal PoF bounds (for given information about the system): **Optimal Uncertainty Quantification!** (OUQ)
Concluding remarks…

Thank you!