Minimum principles for characterizing the trajectories and microstructural evolution of dissipative systems

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Systems with evolving microstructure

• The behavior of the systems of interest is governed by both energy and kinetics, e.g., through an equation of evolution of the form

\[ \partial \Psi(\dot{u}) + DE(t, u) = 0, \]

\[ \begin{align*}
\Psi & \equiv \text{dissipation potential} \\
E & \equiv \text{energy}
\end{align*} \]

• However: Energies of interest often lack differentiability and lower-semicontinuity

• Meaning of equation of evolution, ‘solutions’?

• Effective macroscopic kinetics?

• Wanted: a theoretical framework that extends CoV to dissipative problems...
Classical rate variational problems

- Potential energy:
  \[ E(t, u) = \frac{C}{2} u^2 - f(t)u \]

- Dissipation potential: \( \Psi(v) = \frac{\eta}{2} v^2 \)

- Rate potential:
  \[ G(t, u, v) \equiv \Psi(v) + D E(t, u) v \]

- Rate problem: Given \( t, u, \)
  \[ \min_v G(t, u, v) \]

- Euler-Lagrange equations:
  \[ \partial \Psi(v) + D E(t, u) = 0 \]
Energy-dissipation functionals

- Energy-dissipation functional: For $\epsilon > 0$,
  \[ F_\epsilon(u) = \int_0^T e^{-t/\epsilon} G(t, u, \dot{u}) \, dt \]

- Minimum principle: $\mathbb{Y} = \{ u : [0, T] \rightarrow Y \}$,
  \[ \inf_{u \in \mathbb{Y}} F_\epsilon(u) \]

- Euler-Lagrange eqs., $G(u, \dot{u}) = \Psi(\dot{u}) + DE(u)\dot{u}$,
  \[ -\epsilon D^2\Psi(\dot{u})\ddot{u} + D\Psi(\dot{u}) + DE(t, u) = 0 \]

- Relaxation: $\text{sc}^{-1} F_\epsilon(u) \overset{?}{=} \int_0^T e^{-t/\epsilon} \bar{G}_\epsilon(t, u, \dot{u}) \, dt$

- Causal limit: $\bar{G}_\epsilon \rightarrow \bar{G}$ as $\epsilon \rightarrow 0$
LEFM energy-dissipation functionals

- Energy: $E(u) = \int_{\Omega} W(\nabla u) \, dx + \text{forcing terms}$
- Dissipation: $\psi(v) = \int_{\Gamma^{n-2}} \psi(v) \, d\mathcal{H}^{n-2}$
- Crack front $F$, velocity $v$, defined distributionally
The rate problem of LEFM

Crack-tip equation of motion for fatigue crack growth

Crack-growth data for 2024-T3 aluminum alloy
(P. Paris and F. Erdogan, ASME Trans (1963)
The rate problem of LEFM

Crack-growth data for 2024-T3 aluminum alloy
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LEFM energy-dissipation functionals

- Dissipation: \[ \Psi = \int_{F(t)} \left( \alpha + v^p \right) d\mathcal{H}^{n-2} \]

  nucleation energy
d  rate-dependent crack-tip equation of motion

- Energy-dissipation functional: \[ F_\epsilon(u) := \int_0^T e^{-t/\epsilon} \left\{ \int_{F(t)} (\alpha + v^p) d\mathcal{H}^{n-2} + \frac{1}{\epsilon} \int_\Omega W(\nabla u) \, dx \right\} \, dt \]

- Trajectories: \( Y \sim \{ u(t) \in SBV_p(\Omega), \text{crack increasing} \} \)

- Variational problem: \( \inf_{u \in Y} F_\epsilon(u) \)
LEFM energy-dissipation functionals

Theorem (C. Larsen, MO, C.L. Richardson) *The lower semicontinuous envelop of $F_\epsilon$ in $\mathbb{Y}$ is:*

$$sc^{-} F_\epsilon(u) =$$

$$\int_0^T e^{-t/\epsilon} \left\{ \frac{1}{\epsilon} \int_\Omega W(\nabla u) \, dx + \gamma \int_{F(t)} v \, d\mathcal{H}^{n-2} \right\} \, dt$$

where: $\gamma = p \left( \frac{\alpha}{p - 1} \right)^{\frac{p-1}{p}}$

- Relaxed energy-dissipation functional is **rate-independent**! (Griffith brittle fracture)
LEFM energy-dissipation functionals

**Sketch of proof:** Mother-daughter mechanism:

- Twin daughters (optimal by Jensen’s inequality):

\[
\psi = n\alpha + n \left( \frac{v}{n} \right)^p \rightarrow \min \Rightarrow
\]

\[
n_{\min} = \left( \frac{p - 1}{\alpha} \right)^{1/p} v, \quad \psi_{\min} = p \left( \frac{\alpha}{p - 1} \right)^{(1-1/p)} v
\]
LEFM energy-dissipation functionals

• The mother-daughter crack mechanism:

(Ortiz, *IJSS*, 1988)

(F. Abraham, M. Buehler, H. Gao...)
Concluding remarks

- Energy-dissipation functionals provide a useful tool for understanding microstructure evolution within the framework of the calculus of variations.
- They help to identify the ‘effective’ kinetics and energetics of systems that exhibit evolving microstructure.
- Recovery sequences yield insight into microstructural evolution mechanisms.
- Mother-daughter mechanism beats crack-front rate-dependency in fracture mechanics.
- Causal limit?
- Inertia?